CONCURSUL NAȚIONAL DE MATEMATICĂ APLICATĂ

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ETAPA LOCALĂ

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CLASA a IX-a

H2 **Filiera Teoretică: Profilul Real – Specializarea Științe ale naturii**

BAREM DE CORECTARE ȘI NOTARE

1. (7p) Să se determine toate perechile de numere reale $(x\ ;\ y)$ care verifică simultan relațiile: $|x-2|+|y-5|=2$ și $y=3+|x-2|$

Rezolvare:

$y=3+\|x-2\| \Rightarrow \|x-2\|=y-3 \Rightarrow \begin{cases} y-3\geq 0 \Rightarrow y\geq 3 \\ x=y-1 \ sau \ x=5-y \end{cases}$	2p
$y-3+\|y-5\|=2 \Rightarrow \|y-5\|=-y+5 \Rightarrow \begin{cases} -y+5\geq 0 \Rightarrow y\leq 5 \\ y=5 \ sau \ (\forall) y\in \mathbb{R} \end{cases} \Rightarrow y\leq 5$	2p
$y\in[3,\ 5] \Rightarrow 0\leq y-3\leq 2 \Rightarrow \|x-2\|\leq 2 \Rightarrow -2\leq x-2\leq 2 \Leftrightarrow x\in[0,\ 4]$	2p
$S=\{(x,\ y)/\, x\in[0,\ 4],\ y\in[3,\ 5]\}$ sau $\{(y-1,\ y),(5-y,\ y)/\, y\in[3,\ 5]\}$	1p

2. Dacă $a_n=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+....................+\frac{1}{1+2+...........+n}\right)$, $n\in \mathbb{N}^*$, atunci să se arate că:

a) **(3p)** $\sum_{k=1}^{n} a_k^2-a_{k+1}^2<\frac{1}{4}$

b) **(4p)** $\sum_{k=1}^{n}\left[\frac{1}{(k+1)^2(k+2)^2}-\frac{2}{(k+2)^2}\right]\cdot\frac{1}{a_k^2}=\frac{1}{2}\left[\frac{1}{n+1}+\frac{1}{n+2}-\frac{3}{2}\right]$

Rezolvare:

a) $a_n=\frac{1}{2}\left(\frac{2}{1\cdot 2}+\frac{2}{2\cdot 3}+...........+\frac{2}{n(n+1)}\right)=1-\frac{1}{n+1}=\frac{n}{n+1}$	2p
$\sum_{k=1}^{n}(a_k^2-a_{k+1}^2)=a_1^{\ 2}-a_{n+1}^2=\frac{1}{4}-\left(\frac{n+1}{n+2}\right)^2<\frac{1}{4}.$	1p
b) $\frac{a_k^2-a_{k+1}^2}{a_k^2}=1-\left(\frac{a_{k+1}}{a_k}\right)^2=1-1-\frac{1}{k(k+2)}=-\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)$	2p

$\sum_{k=1}^{n}\left[\frac{1}{(k+1)^2(k+2)^2}-\frac{2}{(k+2)^2}\right]\cdot\frac{1}{a_k^2}=\sum_{k=1}^{n}\left[\frac{a_k^2-a_{k+1}^2}{a_k^2}\right]=-\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right)$	1p
$=-\frac{1}{2}\left[\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right]=\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}-\frac{3}{2}\right).$	1p

3. Fie $b_n=\frac{1}{p}+\frac{1}{p^2}+................+\frac{1}{p^n}$, $n\in N*$, unde p - număr natural nenul. Să se arate că :

a) **(5p)** $\sum_{k=1}^{n}\frac{p^k-1}{p^{k+1}-1}\cdot p\cdot b_{k+1}>\frac{n}{p^n}$. , $p\neq 1$

b) **(2p)** Dacă $p=1$ atunci : $\sum_{k=1}^{n}\frac{p^{k-1}+p^{k-2}+........+p+1}{p^k+p^{k-1}+........+p+1}\cdot p\cdot b_{k+1}=\frac{n(n+1)}{2}$

Rezolvare:

a) $b_n=\frac{1}{p}+\frac{1}{p^2}+............+\frac{1}{p^n}\quad\Big\vert\cdot\frac{1}{p}$ $\frac{1}{p}b_n=\frac{1}{p^2}+\frac{1}{p^3}+........+\frac{1}{p^{n+1}}\Rightarrow b_n\left(\frac{1}{p}-1\right)=\frac{1}{p^{n+1}}-\frac{1}{p}\Rightarrow b_n=\frac{p^n-1}{p^n(p-1)}$	2p
$\frac{b_k}{b_{k+1}}=\frac{p^k-1}{p^k(p-1)}\cdot\frac{p^{k+1}(p-1)}{p^{k+1}-1}=\frac{p(p^k-1)}{p^{k+1}-1}\Rightarrow\sum_{k=1}^{n}\frac{p(p^k-1)}{p^{k+1}-1}\cdot b_{k+1}=\sum_{k=1}^{n}\frac{b_k}{b_{k+1}}\cdot b_{k+1}$	1p
$\sum_{k=1}^{n}b_k=b_1+b_2+...........+b_n=\frac{1}{p}+\left(\frac{1}{p}+\frac{1}{p^2}\right)+\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}\right)+......+\left(\frac{1}{p}+\frac{1}{p^2}+....+\frac{1}{p^n}\right)=$	1p
$=\frac{n}{p}+\frac{n-1}{p^2}+.........+\frac{1}{p^n}>\underbrace{\frac{1}{p^n}+\frac{1}{p^n}+.........+\frac{1}{p^n}}_{n-ori}=\frac{n}{p^n}$	1p
b) $p=1\Rightarrow\sum_{k=1}^{n}\frac{k}{k+1}\cdot(k+1)=\sum_{k=1}^{n}k=1+2+.......+n=\frac{n(n+1)}{2}$	2p

4. (7p) Fie ΔABC în care AM - mediană , BD- bisectoare . Să se arate că :

$$\overline{AM}+\overline{BD}=-\frac{1}{2}\cdot\overline{AB}+\frac{a+3c}{2(a+c)}\cdot\overline{AC}.\quad (a=BC,b=AC,c=AB)$$

Rezolvare:

AM- mediană , atunci : $\overline{AM}=\frac{1}{2}\left(\overline{AB}+\overline{AC}\right)$	2p
$\overline{BD}=\frac{a}{a+c}\overline{BA}+\frac{c}{a+c}\overline{BC}=-\frac{a}{a+c}\overline{AB}+\frac{c}{a+c}\left(\overline{AC}-\overline{AB}\right)=-\overline{AB}+\frac{c}{a+c}\overline{AC}$	3p
$\overline{AM}+\overline{BD}=-\frac{1}{2}\cdot\overline{AB}+\frac{a+3c}{2(a+c)}\cdot\overline{AC}$	2p

Notă: Orice altă soluţie corectă se va puncta corespunzător.